where

$$a_1 = b_1 = 1, \quad a_{n+1} = (-1)^n \prod_{i=1}^{n} \left(\frac{2i+1}{2i}\right)$$

$$b_{n+1} = \frac{(-1)^{n+1} b_0}{2n-1}, \quad n = 1, 2, 3 \ldots$$

The quotient of these two series[3] is thus of the form

$$\frac{(1+z^2)^{-3/2}}{(1+b_0 \tan^{-1} z)} = c_1 + c_2 z + c_3 z^2 + \ldots c_n z^{n-1} \quad \text{(A7)}$$

where $c_1 = 1$, and for n odd

$$c_{n+1} = -\sum_{i=1}^{(n+1)/2} b_{i+1} c_{n-2(i-1)}$$

and for n even

$$c_{n+1} = a_{\frac{n+2}{2}} - \sum_{i=1}^{n/2} b_{i+1} c_{n-2(i-1)}$$

By using Eq. (A7) in Eq. (A5), the integral I_2 may be written as

$$I_2 = \frac{1}{\omega} \int \frac{(c_1 + c_2 z + c_3 z^2 + \ldots c_n z^{n-1})(Az + B)dz}{Z^{1/2}} \quad \text{(A8)}$$

The integral I_2 may be termwise integrated for the bounded interval $(-\pi/4 < \omega t < \pi/4)$.

The solution for the integral I_2 may be written as

$$I_2 = (1/\omega)[c_1 B K_1 + (c_1 A + c_2 B) K_2 + (c_2 A + c_3 B) K_3 + \ldots (c_{n-1} A + c_n B) K_n] \quad \text{(A9)}$$

where the K's are evaluated[4] at $n = 1, 2, 3, \ldots$ from

$$K_1 = \int \frac{dz}{Z^{1/2}} = \frac{\sinh^{-1}}{\alpha^{1/2}} \left(\frac{2\alpha z + \gamma}{q^{1/2}}\right) \quad \text{(A10)}$$

and $q = 4\alpha\beta - \gamma^2$

$$K_2 = \int \frac{z dz}{Z^{1/2}} = \left(\frac{Z^{1/2}}{\alpha} - \frac{\gamma}{2\alpha} K_1\right) \quad \text{(A11)}$$

The $(n + 1)$th term

$$K_{n+1} = \left[z^{n-1} Z^{1/2} - \frac{(2n-1)\gamma K_n}{2} - (n-1)\beta K_{n-1}\right] \Big/ n\alpha$$

$$n = 2, 3, 4 \ldots \quad \text{(A12)}$$

By using Eq. (A4), the remaining integral I_4 is written as

$$I_4 = \frac{1}{\omega} \int \frac{(Az + B) z dz}{(1+z^2)^{3/2}(1 + b_0 \tan^{-1} z) Z^{1/2}} \quad \text{(A13)}$$

By use of a procedure similar to that used with integral I_2, the solution for integral I_4 is as shown in

$$I_4 = (1/\omega)[c_1 B K_2 + (c_1 A + c_2 B) K_3 + (c_2 A + c_3 B) K_4 + \ldots (c_{n-1} A + c_n B) K_{n+1}] \quad \text{(A14)}$$

where $K_2, K_3, \ldots, K_{n+1}$ are evaluated in Eqs. (A11) and (A12).

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Analysis of Sprays from Rocket Engine Injectors

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OVER the last 15 years, the importance of obtaining uniformity of liquid propellant "mixing" to obtain high combustion efficiency in rocket engines has been shown both analytically and experimentally.[1–6] While these studies have contributed to an understanding of the influence of propellant distribution on combustion efficiency, they did not define the interrelationship between the quality of the over-all propellant mixing and the resulting mixing-limited combustion efficiency. An analytical relationship of this type is needed in preliminary design of rocket engine injectors to compare analytically the effect of spray quality on combustion efficiency for various propellant combinations. It should be pointed out that atomization is also an important parameter limiting the performance obtained in most rocket engines. This paper, however, considers only mixing effects on combustion efficiency.

Mixing and combustion occur in three principal zones: 1) in the prereaction zone the propellants are atomized into sprays and mixed; 2) vaporization and combustion are initiated; and 3) as combustion proceeds, the resulting gas accelerates the flow down the chamber in a stream tube manner. While some transverse mixing occurs in the combustion zones, experimental data show that the combustion performance calculated from the mass and mixture ratio (MR) distribution at the end of the prereaction zone agree quite well with that actually obtained.[7]

This Note describes an analytical model defining the relationship between the propellant spray mixing quality determined using cold-flow measurements (see Ref. 7 for details) and combustion performance. Experimental verification of the results predicted from cold-flow analysis is also described. Comparisons are then made of the effect of mixing uniformity on the c^* mixing limited performance for various propellant combinations.

Mixing Quality

Following Rupe[1] we define the mixing quality as the sum of the mass-weighted deviations in mixture ratio from the injected mixture ratio. Rupe considers the cumulative distribution plot for a nonuniform mixture and a monodisperse mixture as shown in Fig. 1. (Monodisperse mixture is here defined as a bipropellant mixture having completely constant mixture ratio in both space and time.) In Fig. 1, $r \equiv MR/(1 + MR)$ at any point in the mixture, and R is the value of r corresponding to the injected mixture ratio. With these definitions, the mass having $r < R$ is to the left and the mass having $r > R$ is to the right of the monodisperse line ($r \equiv R$).

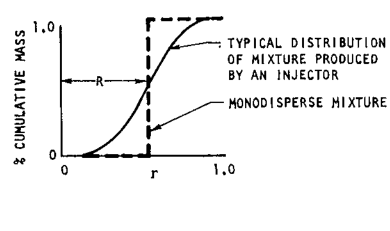

Fig. 1 Normalized mass and mixture ratio distribution plot.

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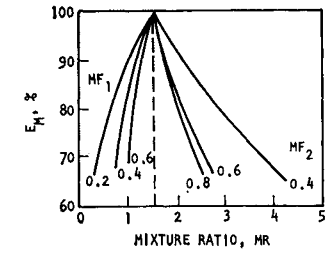

Fig. 2 Solution of Eq. (4) for a two-stream-tube model; injected mixture ratio = 1.6.

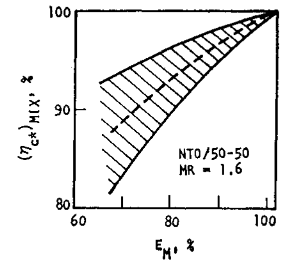

Fig. 4 Relationship between E_m and $(\eta_c{}^*)$ for a mixture ratio range of from 0.1 to 100; propellants NTO/50-50.

The average value of the mass in each of these regions, designated (a) and (b), is determined as follows:

$$\underset{a)\ r < R}{\int_0^{MF_R} (R - r)dMF \Big/ \int_0^R dr}; \quad \underset{b)\ r > R}{\int_{MF_R}^1 (R - r)dMF \Big/ \int_1^R dr} \tag{1}$$

where MF = mass fraction of oxidizer plus fuel at any point. Averages taken in this manner simply convert the integrated area between the actual distribution curve and the monodisperse mixture to an equivalent rectangular area having one side equal to R and $1 - R$ corresponding to the first and second expressions shown, respectively.

Rupe defines a term E_m, which is 1 minus the sum of the two expressions, i.e.,

$$E_m = 1 - [\text{Eq. (1a)} + \text{Eq. (1b)}] \tag{2}$$

From integration of Eq. (2), it is easily shown that the most uniform mixture corresponds to an equivalent average mass of 100%, and the worst distribution would correspond to an average mass of zero.

Since distribution data are actually taken at definite locations across the spray field, then it is more convenient to calculate E_m using a difference equation rather than the integral form of Eq. (2), i.e.;

$$E_m = 1 - \underbrace{\Sigma MF_i(R - r_i)/R}_{r < R} - \underbrace{\Sigma MF_i(R - r_i)/(R - 1)}_{r > R} \tag{3}$$

A generalized set of solutions to Eq. (3) is possible for an arbitrary number of stream tubes. However, the number of solutions possible can be quite large depending on the number of stream tubes selected. The following derivation is for a two-stream-tube model. It should be remembered, however, that any number of stream tubes could be used and all possible solutions still determined.

For a two-tube model, Eq. (3) reduces to:

$$E_m = 1 - \underbrace{MF_1\,[(R - r_1)/R]}_{r_1 < R} - \underbrace{MF_2\,[(R - r_2)/(R - 1)]}_{r_2 > R} \tag{4}$$

For this example, a spray at an overall MR of 1.6 (R = 0.615) was selected. Once R is designated, then specifying MF_1 and r_1 is sufficient to specify MF_2 and r_2. Thus for various values of the independent variables R, MF_1, and r_1, solutions were obtained for E_m. The solutions to Eq. (4), at the specified overall MR are presented in Fig. 2. To obtain high E_m, deviations in MR must be kept small. This is contrasted with the low values of distribution which show that the lower the spray quality the further the mass must be distributed from the overall MR.

Stream-Tube Model $(\eta_c{}^*)_{\text{mix}}$

If liquid-phase mixing is not uniform within the combustion chamber, then regions exist that are fuel or oxidizer rich. Then, since secondary mixing processes are small, the combustion efficiency developed by a fuel- or oxidizer-rich region will be largely dependent upon the initial mixture ratio in that region. Wrobel[8] has described an analysis of mixing losses whereby the combustion chamber cross section is divided into "i" stream tubes (the geometry of the tubes is arbitrary) each containing propellant at some MR. No mass or energy crosses stream-tube boundaries. At each axial station, along the length of the chamber, the static pressure is uniform for all stream tubes. The resulting equation relating the mixing limited characteristic velocity (c^*) efficiency to the local mass and MR distribution is:[8]

$$(\eta_c{}^*)_{\text{mix}} = \Sigma_i\, MR_i\, c^*{}_i/c^*{}_{\text{theo}} \tag{5}$$

For any given propellant mixture ratio distribution, Eq. (5) provides a simple means of determining c^* efficiency loss due to "mixing."

A generalized graphical solution to Eq. (5) can be obtained for a specified overall MR. This was done for NTO/50-50 (50% UDMH + 50% hydrazine) propellants at an over-all MR of 1.6 (Fig. 3). The characteristic shapes of the curves (see Fig. 3) are similar to those of mixing quality. It is obvious that even minor amounts of mass distributed at MR's other than the overall injected value result in a loss in c^* efficiency.

Relationship of E_m to $(\eta_c{}^*)_{\text{mix}}$

From Eqs. (4) and (5), it is easily shown that the mixing-limited c^* efficiency for a given propellant combination is functionally related to mixing quality in the following manner:

$$(\eta_c{}^*)_{\text{mix}} = f(E_m, MR) \tag{6}$$

Mixture ratio is a parameter in the above relationship, because for a given E_m, the mixing-limited c^* efficiency is still dependent on the injected mixture ratio. With an analytical

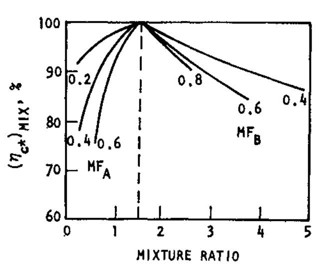

Fig. 3 Solutions of Eq. (7) for a two-stream-tube model; injected mixture ratio = 1.6, propellants = NTO/50-50.

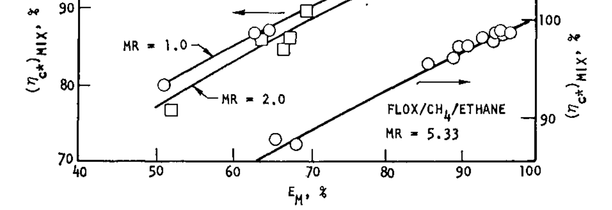

Fig. 5 Comparison of experimental distribution data with the analytically determined mean-line relationship between E_m and $(\eta_c{}^*)_{\text{mix}}$.

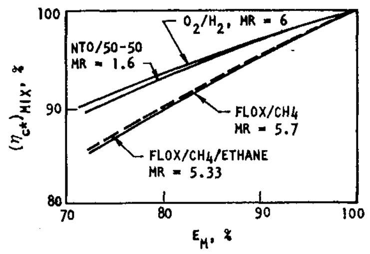

Fig. 6 Comparison of mixing limited C^* performance as a function of mixing uniformity for several propellant combinations.

relationship for E_m and mixing-limited c^* efficiency defined, then, for any desired propellant combination, the E_m required to obtain a specified c^* efficiency can be analytically determined.

We must define the allowable range in MR. The major portion of the spray from typical injectors lies in the range $0.1 \leq MR \leq 100$. Utilizing this result, the relationship between variables of Eq. (6) can be determined. This has been done for NTO/50-50 in Fig. 4. The band represents all possible solutions for $(\eta_{c^*})_{\text{mix}}$ and E_m lying between $MR = 0.1$ and $MR = 100$ for a constant overall MR of 1.6. The dashed line represents the average or mean line through the band. It is obvious from this plot that no unique or even narrow band of possible solutions exist between E_m and $(\eta_{c^*})_{\text{mix}}$ for the selected range in MR, but increasing the number of stream tubes and restricting the solutions to correspond to distributions from specific injectors considerably narrows this band.

Experimental data were utilized to determine if sprays produced by injectors would result in distributions in a narrow band around the mean line determined from the two-tube model. The mean lines for MR's of 1.0 and 2.0 utilizing NTO/50-50 propellants are superimposed on actual measured spray distribution data in Fig. 5, obtained utilizing several impinging stream injectors. Note that, over the range of E_m from about 50 to 95%, the data fall very close ($\pm 1.5\%$) to the analytically determined mean value line. Identical results have also been obtained for FLOX/CH_4/ethane propellants at $MR = 5.33$. These results also are presented in Fig. 5. Note that the data agree to within $\pm 1.0\%$ with the analytically determined mean line. Due to the excellent agreement found, the curves of the type shown in Fig. 5 can now be used to determine the influence of spray quality on performance for differing propellant combi nations.

Comparison for Several Propellant Combinations

The model just described was used to generate predictions of E_m effects on $(\eta_{c^*})_{\text{mix}}$ for four propellant combinations giving the mean line results plotted in Fig. 6. Note that O_2/H_2 is least affected by nonuniformities in mixing, while the FLOX/CH_4 plus ethane propellant are most sensitive. The NTO/50-50 propellant combination is close to the O_2/H_2 curve. The deviations in η_{c^*} for a given E_m is due to the shape of the c^* vs MR curve for each propellant.

Conclusions

A rather simple relationship between mixing uniformity E_m and mixing-limited c^* efficiency $(\eta_{c^*})_{\text{mix}}$ can be used to predict actual c^* mixing-limited performance. This relationship allows a prediction of the sensitivity to mixing of any propellant combination on η_{c^*} to be analytically determined.

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Conical Nozzle Flow in the Rarefied Regime

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Nomenclature

A	=	cross-sectional area
h,h_t	=	enthalpy and total enthalpy, respectively
k	=	coefficient of thermal conductivity
L	=	length of nozzle
$\dot{m},\bar{m}$	=	mass flow rate and dimensionless values
$\dot{N},\bar{N}$	=	molecular flow rate and dimensionless value [Eq. (19)]
N	=	number of points in grid station
K_n,K_n'	=	Knudsen number and function of K_n, respectively
P_r,P_e	=	Prandtl and Reynolds numbers
p	=	pressure
q	=	heat flux
R_g	=	gas constant
r	=	radius coordinate in spherical coordinates, Fig. 1
T,t	=	temperature and time, respectively
u,u_s	=	dimensionless velocity and slip velocity, respectively
v	=	velocity when subscripted, otherwise dimensionless
$\bar{V}$	=	mean molecular velocity
z	=	nozzle longitudinal coordinate
β	=	integral limit, [Eq. (18)]
γ	=	ratio of specific heats
η,λ	=	molecular density and mean free path, respectively
θ	=	angle coordinate
θ_m	=	nozzle half angle
μ,ρ	=	viscosity and density, respectively
τ	=	shear stress
ϕ	=	angle coordinate

Subscripts

e,i	=	nozzle exit and entrance, respectively
m,n	=	grid coordinate integers

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